On the Ratio between Sector and Triangle in the Orbit of a Celestial Body.

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1. THIS ratio may be expressed by the formula

$$\frac{1}{\eta}=\frac{rr'\sin(v'-v)}{\sqrt{p}\,\tau}=\frac{\sin 2\theta}{2\theta}, \tag{1}$$

where the mass of the body is neglected, r and r' are the radii vectores, v and v' the corresponding true anomalies, p the semi-parameter, and τ the product of the intervening time and the constant of the solar system.

By Taylor's theorem, including terms of the fourth order,

$$v=v_0-\frac{1}{2}\frac{dv_0}{d\tau}\tau+\frac{1}{8}\frac{d^2v_0}{d\tau^2}\tau^2-\frac{1}{48}\frac{d^3v_0}{d\tau^3}\tau^3+\frac{1}{384}\frac{d^4v_0}{d\tau^4}\tau^4-\ldots,$$

$$v'=v_0+\frac{1}{2}\frac{dv_0}{d\tau}\tau+\frac{1}{8}\frac{d^2v_0}{d\tau^2}\tau^2+\frac{1}{48}\frac{d^3v_0}{d\tau^3}\tau^3+\frac{1}{384}\frac{d^4v_0}{d\tau^4}\tau^4+\ldots,$$

where v_0 is the true anomaly corresponding to the mean of the times;

$$\therefore\ v'-v=\frac{dv_0}{d\tau}\tau+\frac{1}{24}\frac{d^3v_0}{d\tau^3}\tau^3+\ldots.$$

$$\therefore\ \sin(v'-v)=\frac{dv_0}{d\tau}\tau+\left(\frac{1}{24}\frac{d^3v_0}{d\tau^3}-\frac{1}{6}\left(\frac{dv_0}{d\tau}\right)^3\right)\tau^3.$$

The well-known expressions $\frac{dv_0}{d\tau}=\frac{\sqrt{p}}{r_0^2}$ and $\frac{p}{r_0}=1+e\cos v_0$ give, by successive differentiation,

$$\frac{d^2r_0}{d\tau^2}=\frac{p-r_0}{r_0^3},$$

$$\frac{d^3v_0}{d\tau^3}=\frac{\sqrt{p}}{r_0^2}\left(\frac{6}{r_0^2}\left(\frac{dr_0}{d\tau}\right)^2-\frac{2}{r_0}\frac{d^2r_0}{d\tau^2}\right),$$

whence by substitution, including terms of the third order,

$$\frac{\sin(v'-v)}{\sqrt{p\tau}} = \frac{1}{r_0^2}\left[1 + \left(\frac{1}{4r_0^2}\left(\frac{dr_0}{d\tau}\right)^2 - \frac{3p-r_0}{12r_0^4}\right)\tau^2 + \ldots.\right]. \tag{2}$$

Developing r and r' in the same manner as v and v' were developed, and multiplying the results, we have

$$rr' = r_0^2\left[1 - \left(\frac{1}{4r_0^2}\left(\frac{dr_0}{d\tau}\right)^2 - \frac{p-r_0}{4r_0^4}\right)\tau^2 + \ldots.\right]. \tag{3}$$

The product of (2) and (3) gives

$$\frac{1}{\eta} = 1 - \frac{\tau^2}{6r_0^3} + \ldots. = 1 - \frac{4\theta^2}{6} + \ldots.;$$

$$\therefore 2\theta = \frac{\tau}{r_0^{\frac{3}{2}}} + \ldots. = \frac{\tau}{(rr')^{\frac{3}{4}}} + \ldots. = \frac{2^{\frac{3}{2}}\tau}{(r+r')^{\frac{3}{2}}} + \ldots..$$

2. For a closer approximation we have the well-known equations

$$\frac{\eta^2}{m} = \frac{1}{l + \sin^2\frac{1}{2}g},$$

$$\frac{\eta^2}{m}(\eta - 1) = \frac{2g - \sin 2g}{\sin^3 g},$$

in which

$$l = \frac{\sin^2\frac{1}{2}\gamma}{\cos\gamma}, \quad m = \frac{\theta_0^2}{2\cos^3\gamma}, \quad \cos\gamma = \frac{2\sqrt{rr'}}{r+r'}\cos\tfrac{1}{2}(v'-v), \quad \theta_0^2 = \frac{2\tau^2}{(r+r')^3}$$

and g is the half difference of the eccentric anomalies.

For the first of these equations we may write

$$x = \sin^2\tfrac{1}{2}g = \frac{m}{\eta^2} - l,$$

or by substitution,

$$x = \frac{1}{8}\frac{\theta_0^2}{\theta^2}\frac{\sin^2 2\theta}{\cos^3\gamma} - \frac{\sin^2\frac{1}{2}\gamma}{\cos\gamma}.$$

In the same manner the second becomes

$$\frac{\theta^2}{\theta_0^2}\cdot\frac{\cos^3\gamma}{\cos^3\theta}\cdot\frac{2\theta - \sin 2\theta}{\sin^3\theta} = \frac{2g - \sin 2g}{\sin^3 g}.$$

Gauss has developed the second member of the latter equation in a series arranged according to the ascending powers of x, as follows:

$$\frac{2g - \sin 2g}{\sin^3 g} = \frac{4}{3}\left(1 + \frac{6}{5}x + \frac{6}{5}\cdot\frac{8}{7}x^2 + \ldots.\right),$$

whence, if we develop $\frac{2\theta - \sin 2\theta}{\sin^3\theta}$ in a similar series arranged according to the ascending powers of $z = \sin^2 \frac{1}{2}\theta$, we shall have

$$\theta^2 \cos^3\gamma \left(1 + \frac{6}{5} z + \ldots\right) = \theta_0^2 \cos^3\theta \left(1 + \frac{6}{5} x + \ldots\right),$$

or, dividing by $\left(1 + \frac{6}{5} z + \ldots\right)\left(1 + \frac{6}{5} x + \ldots\right)$,

$$\theta^2 \cos^3\gamma \left(1 - \frac{6}{5} x + \ldots\right) = \theta_0^2 \cos^3\theta \left(1 - \frac{6}{5} z + \ldots\right).$$

Substituting the values of x and z, and reducing,

$$\theta^2 \cos^2\gamma \left(1 - \frac{4}{5} \sin^2 \tfrac{1}{2}\gamma + \ldots\right) = \theta_0^2 \cos^2\theta \left(1 - \frac{4}{5} \sin^2 \tfrac{1}{2}\theta + \ldots\right),$$

whence, approximately,

$$\theta = \theta_0 \left(\frac{\cos\theta}{\cos\gamma}\right)^{1.2}.$$

This formula includes terms of the same order as those included in Hansen's method, and, if employed in connection with a table giving the logarithms of the ratios between sines and arcs, is rather more convenient.